南京航空航天大学

2012～2013 学年 《 Matrix Theory 》期中考试试题

考试日期： 2012 年 11 月　　试卷类型：　　　试卷代号：

学院		学号		姓名	

题号	1	2	3	4	5	6	7	8	9	10	总分
得分											

参考答案

Part I (必做题, 70 分)

#1. For the given matrix

$$A=\begin{pmatrix}1&3&-2&1\\2&1&3&2\\3&4&5&6\end{pmatrix}$$

(1) find the <u>reduced row echelon form</u>（简化行阶梯形）of A;
(2) find a basis for the row space of A ;
(3) find a basis for the for the column space of A ;
(4) find a basis for the nullspace of A .

Solution:

(1) $$A=\begin{pmatrix}1&3&-2&1\\2&1&3&2\\3&4&5&6\end{pmatrix}\to\begin{pmatrix}1&3&-2&1\\0&-5&7&0\\0&0&4&3\end{pmatrix}\to\begin{pmatrix}1&3&0&5/2\\0&1&0&21/20\\0&0&1&3/4\end{pmatrix}\to\begin{pmatrix}1&0&0&-13/20\\0&1&0&21/20\\0&0&1&3/4\end{pmatrix}=U$$

(2) $(1,0,0,-13/20)$, $(0,1,0,21/20)$, $(0,0,1,3/4)$ form a basis for the row space of A, since row elementary operations do not change the row space of A.

(3) The 1st, 2nd, and 3rd column vectors of U are linearly independent. The 4th column vector is a linear combination of the first three column vectors. Hence, $(1,2,3)^T$, $(3,1,4)^T$, $(-2,3,5)^T$ form a basis for the column space of A.

(4) Solving the system $A\mathbf{x}=\mathbf{0}$, we obtain a basis $(13,-21,-15,20)^T$ for the nullspace of A.

#2. Let $\mathbf{e}_1,\mathbf{e}_2$ be a basis for $\mathbf{R}^2$ and $\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3$ be a basis for $\mathbf{R}^3$,

where $(\mathbf{e}_1,\mathbf{e}_2)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $(\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3)=\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$.

Let L : $\mathbf{R}^2\to\mathbf{R}^3$ be the linear transformation defined by

$$L(\mathbf{x})=x_1\mathbf{b}_1+x_2\mathbf{b}_2+(x_1+x_2)\mathbf{b}_3.$$

Find the matrix A representing L with respect to the bases $[\mathbf{e}_1, \mathbf{e}_2]$ and $[\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$.

Solution:

$$L(\mathbf{e}_1) = \mathbf{b}_1 + \mathbf{b}_3$$

$$L(\mathbf{e}_2) = \mathbf{b}_2 + \mathbf{b}_3$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

#3. Let **S** be the two dimensional subspace of $\mathbf{R}^3$ spanned by

$$\mathbf{u}_1 = (1,0,2)^T \text{ and } \mathbf{u}_2 = (0,1,-2)^T$$

Find a basis for $\mathbf{S}^\perp$ and determine the projection matrix P that projects vectors in $\mathbf{R}^3$ onto $\mathbf{S}^\perp$.

Solution: $\mathbf{x} \in \mathbf{S}^\perp$ if and only if $\mathbf{x} \perp \mathbf{u}_1$ and $\mathbf{x} \perp \mathbf{u}_2$. That is

$$x_1 + 2x_3 = 0$$

$$x_2 - 2x_3 = 0$$

$\mathbf{S}^\perp = \text{span}\{(-2,2,1)^T\}$. An orthonormal basis for $\mathbf{S}^\perp$ is $(-2/3, 2/3, 1/3)^T$

The projection matrix is

$$\begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix} (-2/3, 2/3, 1/3) = \begin{pmatrix} 4/9 & -4/9 & -2/9 \\ -4/9 & 4/9 & 2/9 \\ -2/9 & 2/9 & 1/9 \end{pmatrix}$$

#4. For the given matrix A, find all possible values of the scalar β that make A diagonalizable or show that no such values exist.

$$A = \begin{pmatrix} 4 & 6 & -2 \\ -1 & -1 & 1 \\ 0 & 0 & \beta \end{pmatrix}$$

Solution: The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = \beta$.

If $\beta \neq 1$ and $\beta \neq 2$, then A is diagonalizable since A has 3 distinct eigenvalues, and hence A has 3 independent eigenvectors.

If $\beta = 1$, then 2 is a single eigenvalue and 1 is a double eigenvalue of matrix A. For the double eigenvalue,

$$A - 1 \cdot I = \begin{pmatrix} 3 & 6 & -2 \\ -1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad N(A - 1 \cdot I) = \text{span}\{(-2,1,0)^T\}, \text{ which is of dimension 1.}$$

Hence, in this case A is not diagonalizable since the geometric multiplicity of $\lambda_1 = 1$ is less than its algebraic multiplicity.

If $\beta = 2$, then 1 is a single eigenvalue and 2 is a double eigenvalue of matrix A. For this double eigenvalue,

$$A - 2 \cdot I = \begin{pmatrix} 2 & 6 & -2 \\ -1 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad N(A - 2 \cdot I) = \text{span}\{(1,0,1)^T, (3,-1,0)^T\}, \text{ which is of dimension 2.}$$

Thus, in this case A is diagonalizable since for each eigenvalue of A, its algebraic multiplicity is the same as its geometric multiplicity.

In summary, A is diagonalizable if and only if $\beta \neq 1$.

Part II (选做题, 30 分)

请选择下列三题中的两题解答，并在所选的题号上划圈，否则按得分最低的两题计分.

#5. Let $\mathbf{S}$ be a subspace of $\mathbf{R}^1$. Show that either $\mathbf{S}=\{\mathbf{0}\}$ or $\mathbf{S}=\mathbf{R}^1$.

Proof If $\mathbf{S}\neq\{\mathbf{0}\}$, then there exists a nonzero element $(a)\in\mathbf{S}$, where a is a nonzero real number. For any element $(x)\in\mathbf{R}^1$, $\frac{x}{a}$ is a real number. Thus, $\frac{x}{a}(a)=(x)\in\mathbf{S}$ since $\mathbf{S}$ is closed under scalar multiplication. Thus, any element in $\mathbf{R}^1$ is also in $\mathbf{S}$, and hence $\mathbf{S}=\mathbf{R}^1$.

#6. If $\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3$ are subspaces of vector space $\mathbf{V}$, show that $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$ is a direct sum if and only if

$\mathbf{V}_1\cap(\mathbf{V}_2+\mathbf{V}_3)=\{\mathbf{0}\}$, $\mathbf{V}_2\cap(\mathbf{V}_1+\mathbf{V}_3)=\{\mathbf{0}\}$, and $\mathbf{V}_3\cap(\mathbf{V}_1+\mathbf{V}_2)=\{\mathbf{0}\}$.

(注: 不可利用书中 36 页上的 Theorem 1.7.3)

Proof $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$ is a direct sum if each vector in the sum can be *uniquely* represented as

$$\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3, \text{ where } \mathbf{x}_i\in\mathbf{V}_i\ (i=1,2,3).$$

Suppose that $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$ is a direct sum. If $\mathbf{V}_1\cap(\mathbf{V}_2+\mathbf{V}_3)\neq\{\mathbf{0}\}$, then there is a nonzero element $\mathbf{u}\in\mathbf{V}_1\cap(\mathbf{V}_2+\mathbf{V}_3)$.

$\mathbf{u}$ can be represented as $\mathbf{u}=\mathbf{u}+\mathbf{0}+\mathbf{0}$, where $\mathbf{u}\in\mathbf{V}_1$, and $\mathbf{0}\in\mathbf{V}_2$, $\mathbf{0}\in\mathbf{V}_3$.

And also, $\mathbf{u}$ can be represented as $\mathbf{u}=\mathbf{0}+\mathbf{x}_2+\mathbf{x}_3$ since $\mathbf{u}\in\mathbf{V}_2+\mathbf{V}_3$, where $\mathbf{0}\in\mathbf{V}_1$, $\mathbf{x}_2\in\mathbf{V}_2$, $\mathbf{x}_3\in\mathbf{V}_3$. Thus, $\mathbf{u}$ has two distinct representations. This is a contradiction. Thus, when $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$ is a direct sum, we must have $\mathbf{V}_1\cap(\mathbf{V}_2+\mathbf{V}_3)=\{\mathbf{0}\}$. Similarly, we can prove that $\mathbf{V}_2\cap(\mathbf{V}_1+\mathbf{V}_3)=\{\mathbf{0}\}$, and $\mathbf{V}_3\cap(\mathbf{V}_1+\mathbf{V}_2)=\{\mathbf{0}\}$.

Conversely, suppose that $\mathbf{V}_1\cap(\mathbf{V}_2+\mathbf{V}_3)=\{\mathbf{0}\}$, $\mathbf{V}_2\cap(\mathbf{V}_1+\mathbf{V}_3)=\{\mathbf{0}\}$, and $\mathbf{V}_3\cap(\mathbf{V}_1+\mathbf{V}_2)=\{\mathbf{0}\}$.
If $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$ is not a direct sum, then there exists an element $\mathbf{u}\in\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$, such that

$$\mathbf{u}=\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3 \text{ and } \mathbf{u}=\mathbf{y}_1+\mathbf{y}_2+\mathbf{y}_3, \text{ where } \mathbf{x}_i,\mathbf{y}_i\in\mathbf{V}_i\ (i=1,2,3).$$

and there exist at least one integer $i\in\{1,2,3\}$ such that $\mathbf{x}_i\neq\mathbf{y}_i$. Without loss of generality, assume that $i=1$.

#7. Let σ be a linear transformation on vector space $\mathbf{V}$ over the complex number field, and $\mathbf{S}$ be a σ-invariant subspace of $\mathbf{V}$. Show that there exists a nonzero vector $\mathbf{u}\in\mathbf{S}$, such that $\sigma(\mathbf{u})=\lambda\mathbf{u}$, where λ is a scalar. (For the definition of invariant subspace, see page 92 in the textbook.)

提示: 取 $\mathbf{S}$ 的一组基 $\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_k$，则 $\sigma(\mathbf{u}_1),\sigma(\mathbf{u}_2),\cdots,\sigma(\mathbf{u}_k)$ 都可由 $\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_k$ 的线性组合表示,

利用线性组合的系数矩阵.

Proof In formal multiplication, we can write

$$(\sigma(\mathbf{u}_1),\sigma(\mathbf{u}_2),\cdots,\sigma(\mathbf{u}_k)) = (\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_k)A$$

where A is a $k\times k$ matrix. Let λ be an eigenvalue of A, and $\mathbf{x}$ be an eigenvector such that $A\mathbf{x}=\lambda\mathbf{x}$, where $\mathbf{x}=(x_1,x_2,\cdots,x_k)^T$

Let $\mathbf{u}=x_1\mathbf{u}_1+x_2\mathbf{u}_2+\cdots+x_k\mathbf{u}_k=(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_k)\mathbf{x}$, which is not zero.

Then $\sigma(\mathbf{u})=x_1\sigma(\mathbf{u}_1)+x_2\sigma(\mathbf{u}_2)+\cdots+x_k\sigma(\mathbf{u}_k)=(\sigma(\mathbf{u}_1),\sigma(\mathbf{u}_2),\cdots,\sigma(\mathbf{u}_k))\mathbf{x}$

$$=(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_k)A\mathbf{x}=(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_k)\lambda\mathbf{x}=\lambda(\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_k)\mathbf{x}=\lambda\mathbf{u}$$