南京航空航天大学第一届积分竞赛初赛

1. $\frac{1}{\pi}\int_0^{+\infty}\frac{1}{(1+x^2)(1+x^{2023})}\mathrm{d}x$.

解：作换元$t=\frac{1}{x}$可得

$$\int_0^{+\infty}\frac{1}{(1+x^2)(1+x^{2023})}\mathrm{d}x=-\int_{+\infty}^0\frac{1}{\left(1+\frac{1}{t^2}\right)\left(1+\frac{1}{t^{2023}}\right)}\cdot\frac{1}{t^2}\mathrm{d}t=\int_0^{+\infty}\frac{t^{2023}}{(1+t^2)(1+t^{2023})}\mathrm{d}t$$

则有

$$\begin{aligned}I&=\frac{1}{\pi}\int_0^{+\infty}\frac{1}{(1+x^2)(1+x^{2023})}\mathrm{d}x=\frac{1}{2\pi}\int_0^{+\infty}\frac{1+x^{2023}}{(1+x^2)(1+x^{2023})}\mathrm{d}x\\&=\frac{1}{2\pi}\int_0^{+\infty}\frac{1}{1+x^2}\mathrm{d}x=\frac{1}{2\pi}\cdot\arctan x|_0^{+\infty}=\frac{1}{2\pi}\cdot\frac{\pi}{2}=\frac{1}{4}.\square\end{aligned}$$

注：本题把 2023 改成$\alpha>0$答案均不变.

2. $\int_0^1\frac{1}{\sqrt{4-x^2}}\mathrm{d}x$.

解：带入公式$\int\frac{\mathrm{d}x}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+C$可得

$$\int_0^1\frac{1}{\sqrt{4-x^2}}\mathrm{d}x=\arcsin\frac{x}{2}\Big|_0^1=\arcsin\frac{1}{2}=\frac{\pi}{6}.\square$$

3. $\int_0^{\pi}\sqrt{\sin^3x-\sin^5x}\,\mathrm{d}x$.

解：直接化简可得

$$\begin{aligned}\int_0^{\pi}\sqrt{\sin^3x-\sin^5x}\,\mathrm{d}x&=\int_0^{\pi}\sqrt{\sin^3x(1-\sin^2x)}\,\mathrm{d}x=\int_0^{\pi}\sqrt{\sin^3x\cos^2x}\,\mathrm{d}x\\&=\int_0^{\pi}|\cos x|\,(\sin x)^{\frac{3}{2}}\mathrm{d}x=2\int_0^{\frac{\pi}{2}}(\sin x)^{\frac{3}{2}}\mathrm{d}(\sin x)=2\cdot\frac{2}{5}\,(\sin x)^{\frac{5}{2}}\Big|_0^{\frac{\pi}{2}}=\frac{4}{5}.\square\end{aligned}$$

4. $\frac{1}{\pi}\int_0^{+\infty}\frac{1}{(1+x^2)^3}\mathrm{d}x$.

解：作换元$x=\tan t$可得

$$\begin{aligned}\int_0^{+\infty}\frac{1}{(1+x^2)^3}\mathrm{d}x&=\int_0^{\frac{\pi}{2}}\frac{1}{(1+\tan^2t)^3}\cdot\sec^2t\,\mathrm{d}t=\int_0^{\frac{\pi}{2}}\frac{1}{\sec^4t}\mathrm{d}t=\int_0^{\frac{\pi}{2}}\cos^4t\,\mathrm{d}t\\&=\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{3\pi}{16}.\square\end{aligned}$$

5. $\int_0^{+\infty}x\,\mathrm{e}^{-3x}\mathrm{d}x$.

解：由分部积分可得

$$\int_0^{+\infty} x e^{-3x} dx = -\frac{1}{3}\int_0^{+\infty} x d(e^{-3x}) = -\frac{1}{3} x e^{-3x}\Big|_0^{+\infty} + \frac{1}{3}\int_0^{+\infty} e^{-3x} dx$$
$$= -\frac{1}{9} e^{-3x}\Big|_0^{+\infty} = \frac{1}{9}.\square$$

或者作换元$t = 3x$可得

$$\int_0^{+\infty} x e^{-3x} dx = \frac{1}{9}\int_0^{+\infty} t e^{-t} dt = \frac{1}{9}\Gamma(2) = \frac{1}{9}\cdot 1 = \frac{1}{9}.\square$$

或者由$\left[-\frac{1}{9}(3x+1)e^{-3x}\right]' = x e^{-3x}$可得

$$\int_0^{+\infty} x e^{-3x} dx = -\frac{1}{9}(3x+1)e^{-3x}\Big|_0^{+\infty} = \frac{1}{9}.\square$$

6. $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{\sin^2 x\cos^2 x} dx$.

解：利用1的变形可得

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{\sin^2 x\cos^2 x} dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^2 x + \cos^2 x}{\sin^2 x}\cdot\frac{1}{\cos^2 x} dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \left(1 + \frac{1}{\tan^2 x}\right) d(\tan x)$$
$$= \left[\tan x - \frac{1}{\tan x}\right]\Big|_{\frac{\pi}{8}}^{\frac{\pi}{4}} = 2.\square$$

7. $\int_0^2 x^2\sqrt{2x - x^2} dx$.

解：变形可得

$$\int_0^2 x^2\sqrt{2x-x^2} dx = \int_0^2 x^2\sqrt{1-(x-1)^2} dx = \int_{-1}^1 (t+1)^2\sqrt{1-t^2} dt$$
$$= \int_{-1}^1 t^2\sqrt{1-t^2} dt + 2\int_{-1}^1 t\sqrt{1-t^2} dt + \int_{-1}^1 \sqrt{1-t^2} dt$$

利用对称性可得

$$I = \int_{-1}^1 t^2\sqrt{1-t^2} dt + \int_{-1}^1 \sqrt{1-t^2} dt$$

作换元$t = \cos x$可得

$$I = \int_{-1}^1 t^2\sqrt{1-t^2} dt + \int_{-1}^1 \sqrt{1-t^2} dt = \int_0^{\pi} \cos^2 x\sin^2 x dx + \int_0^{\pi} \sin^2 x dx$$
$$= \frac{1}{8}\int_0^{2\pi} \sin^2 x dx + 2\int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{5}{2}\int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{5}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{8}.\square$$

8. $\int_0^{\frac{\pi}{4}} \sec^4 x \, dx$.

解： 变形可得

$$\int_0^{\frac{\pi}{4}} \sec^4 x \, dx = \int_0^{\frac{\pi}{4}} \sec^2 x \cdot \sec^2 x \, dx = \int_0^{\frac{\pi}{4}} \frac{\sin^2 x + \cos^2 x}{\cos^2 x} d(\tan x) = \int_0^{\frac{\pi}{4}} (1 + \tan^2 x) d(\tan x)$$
$$= \left[\tan x + \frac{1}{3}(\tan x)^3\right]\Big|_0^{\frac{\pi}{4}} = \frac{4}{3}.\square$$

9. $\int_0^1 x^3 \ln x \, dx$.

解： 我们直接考虑一般情况 $\int_0^1 x^m (\ln x)^n dx$，有

$$\int_0^1 x^m (\ln x)^n dx = \frac{1}{m+1}\int_0^1 (\ln x)^n d(x^{m+1}) = -\frac{n}{m+1}\int_0^1 x^m (\ln x)^{n-1} dx$$
$$= -\frac{n}{m+1} \cdot (-1) \cdot \frac{n-1}{m+1}\int_0^1 x^m (\ln x)^{n-1} dx$$
$$= \cdots\cdots = \frac{(-1)^n n!}{(m+1)^n}\int_0^1 x^m dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

带入本题可得 $\int_0^1 x^3 \ln x \, dx = \frac{(-1)^1 \cdot 1!}{(3+1)^{1+1}} = -\frac{1}{16}.\square$

10. $\int_0^{\frac{3}{2}} \frac{4}{4-x^2} dx$.

解： 带入公式 $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$ 可得

$$\int_0^{\frac{3}{2}} \frac{4}{4-x^2} dx = 4 \cdot \frac{1}{2 \cdot 2} \ln\left|\frac{2+x}{2-x}\right|\Big|_0^{\frac{3}{2}} = \ln 7.\square$$

11. $\int_0^1 \ln^2 x \, dx$.

解： 作换元 $t = \ln x$ 可得

$$\int_0^1 \ln^2 x \, dx = \int_{-\infty}^0 t^2 e^t dt = \int_0^{+\infty} t^2 e^{-t} dt = \Gamma(3) = 2! = 2.\square$$

12. $\int_0^{2\pi} \frac{\sqrt{2}}{1+\cos^2 x} dx$.

解：作变换可得

$$\int_0^{2\pi}\frac{\sqrt{2}}{1+\cos^2 x}\mathrm{d}x=4\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{\sin^2 x+\cos^2 x}{\sin^2 x+2\cos^2 x}\mathrm{d}x=4\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{\tan^2 x+1}{\tan^2 x+2}\mathrm{d}x$$

作换元$t=\tan x$可得$\mathrm{d}t=\dfrac{1}{\cos^2 x}\mathrm{d}x=(1+\tan^2 x)\mathrm{d}x\Longleftrightarrow \mathrm{d}x=\dfrac{1}{1+t^2}\mathrm{d}t$，则

$$I=4\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{\tan^2 x+1}{\tan^2 x+2}\mathrm{d}x=4\sqrt{2}\int_0^{+\infty}\frac{t^2+1}{t^2+2}\cdot\frac{1}{t^2+1}\mathrm{d}t=4\sqrt{2}\int_0^{+\infty}\frac{1}{t^2+2}\mathrm{d}t$$

带入公式$\displaystyle\int\frac{\mathrm{d}x}{a^2+x^2}=\frac{1}{a}\arctan\frac{x}{a}+C$可得

$$4\sqrt{2}\int_0^{+\infty}\frac{1}{t^2+2}\mathrm{d}t=4\sqrt{2}\cdot\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}\bigg|_0^{+\infty}=4\cdot\frac{\pi}{2}=2\pi.\square$$

13. $\displaystyle\int_0^{\pi}x\sin^5 x\,\mathrm{d}x$.

解：利用公式$\displaystyle\int_0^{\pi}xf(\sin x)\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi}f(\sin x)\mathrm{d}x=\pi\int_0^{\frac{\pi}{2}}f(\sin x)\mathrm{d}x$可得

$$\int_0^{\pi}x\sin^5 x\,\mathrm{d}x=\pi\int_0^{\frac{\pi}{2}}\sin^5 x\,\mathrm{d}x=\pi\cdot\frac{4}{5}\cdot\frac{2}{3}=\frac{8\pi}{15}.\square$$

14. $\displaystyle\int_0^4\frac{x+2}{\sqrt{2x+1}}\mathrm{d}x$.

解：变形可得

$$\int_0^4\frac{x+2}{\sqrt{2x+1}}\mathrm{d}x=\frac{1}{2}\int_0^4\frac{2x+1+3}{\sqrt{2x+1}}\mathrm{d}x=\frac{1}{2}\int_0^4\sqrt{2x+1}\,\mathrm{d}x+\frac{3}{2}\int_0^4\frac{1}{\sqrt{2x+1}}\mathrm{d}x$$
$$=\frac{1}{2}\cdot\frac{1}{3}(2x+1)^{\frac{3}{2}}\bigg|_0^4+\frac{3}{2}(2x+1)^{\frac{1}{2}}\bigg|_0^4=\frac{22}{3}.\square$$

15. $\displaystyle\lim_{n\to\infty}\left[\frac{n}{(2n+1)^2}+\frac{n}{(2n+2)^2}+\cdots+\frac{n}{(2n+n)^2}\right]$.

解：利用定积分定义可得

$$\lim_{n\to\infty}\left[\frac{n}{(2n+1)^2}+\frac{n}{(2n+2)^2}+\cdots+\frac{n}{(2n+n)^2}\right]=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{(2n+i)^2}$$
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{n^2}{(2n+i)^2}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(2+\frac{i}{n}\right)^2}=\int_0^1\frac{1}{(x+2)^2}\mathrm{d}x=-\frac{1}{x+2}\bigg|_0^1=\frac{1}{6}.\square$$