Mid-term Exam of Matrix Theory

Part I

Required Questions $(4 \times 15' = 60')$.

Q1. 1) Let $A = \begin{pmatrix} 2 & -3 & 4 \\ 4 & -6 & 8 \\ 6 & -7 & 8 \end{pmatrix}$,

1. Calculate the characteristics polynomial and eigenvalues of A.
2. Find the determinant divisors, invariant divisors and elementary divisors of A.

2) Given $B = \begin{pmatrix} 17 & -6 \\ 45 & -16 \end{pmatrix}$ and $C = \begin{pmatrix} 14 & -60 \\ 3 & -13 \end{pmatrix}$, please determine if B and C are similar or not. And prove your conclusion.

Q2. Denote $V = \{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in R^{2\times 2} | a_{11} = a_{22}\}$. For any $X \in V$, let $T(X) = PX + XP$, where $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

1. Find a basis of V and show the dimension.
2. Arbitrarily given $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ in V, define

   $$(A, B) = a_{11}b_{11} + 2a_{12}b_{12} + a_{21}b_{21}.$$

   Please show that (A, B) is an inner product on V.
3. Given an orthonormal basis of V under the inner product of 2.
4. Prove that T is a linear transformation on V, and show the matrix representation of T with respect to the basis given in item 1.

Q3. Consider the inner product space $C[-1, 1]$ with inner product defined as

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx, \quad \forall f(x), g(x) \in C[-1, 1].$$

1. Show that 1 and $3x^2 - 1$ are orthogonal.
2. Determine $\|1\|$ and $\|3x^2 - 1\|$.
3. Let $S = \mathbf{L}\{1, 3x^2 - 1\}$ be a subspace of $R[x]_3$, find the optimal approximation of x over S.

Q8. Denote $R[x]_3$ to be the vector space of zero and polynomials with degree less than 3.

1. Determine the dimension of $R[x]_3$ and give a basis of $R[x]_3$.
2. Define the linear transformation $\mathbf{D}$ on $R[x]_3$,

$$\mathbf{D}(f(x)) = f'(x), \quad \forall f(x) \in R[x]_3.$$

   Please give the matrix representation of $\mathbf{D}$ with respect to the basis given in the above item. Show $R(\mathbf{D})$ and $\ker(\mathbf{D})$.

3. Prove that $\mathbf{D}$ is not diagonalizable.
4. Define the inner product on $R[x]_3$,

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx, \quad \forall f(x), g(x) \in R[x]_3,$$

   please Gram-Schmidt orthogonalize the basis given in item 1.

Part II

Preferential Questions $(2 \times 20' = 40')$.

Q5. For any $x \in R^n$, several definitions are given as follows,

$$\|x\|_0 = \sum_{x_i \neq 0} |x_i|^0, \quad \|x\|_p = (\sum_{i=1}^{m} |x_i|^p)^{\frac{1}{p}} \ (0 < p < 1), \quad \|x\|_1 = \sum_{i=1}^{m} |x_i|. \tag{1}$$

1. Please determine if $\|x\|_0$, $\|x\|_p$ and $\|x\|_1$ are valid vector norms or not. And try to defense your decision.
2. Especially when $n = 2$, plot the curves of $\|x\|_0 = 1$, $\|x\|_p = 1$ and $\|x\|_1 = 1$ respectively.

Q6. Given $A \in R^{n\times n}$, summarize the necessary and sufficient conditions of A to be diagonalizable, and prove at least one of them. Determine if the matrix A given in Q1 is diagonalizable or not. If yes, please explain why, if not, please give the Jordan canonical form of A.

Q7. Given $A \in R^{n\times n}$, denote $W = \{X \in R^{n\times n} | AX = XA\}$.

1. Show that W is a subspace of $R^{n\times n}$.
2. Denote
$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$
where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are different from each other. If $A = D$, please determine the dimension of W.
3. If A is similar to D defined as in item 2, please prove that any $X \in W$ is diagonalizable.
4. Given some $X \in W$, if X and A are both diagonalizable, then there exists a nonsingular matrix $P \in R^{n\times n}$ such that $P^{-1}XP$ and $P^{-1}AP$ are diagonal simultaneously.