2013年矩阵论期中测验选做题参考答案

第五题 Definition: Let $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ be subspaces of vector space $\mathbf{V}$. $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$ is a direct sum if each vector $\mathbf{x} \in \mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$ can be uniquely represented as $\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3$, where $\mathbf{x}_k \in \mathbf{V}_k$ for $k=1,2,3$.

Show that $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$ is a direct sum if and only if

$$\dim(\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3)=\dim(\mathbf{V}_1)+\dim(\mathbf{V}_2)+\dim(\mathbf{V}_3)$$

(注: 不可利用书中 36 页上的 Theorem 1.7.3)

Proof

If one of $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ is a zero subspace, then the statement is true by Theorem 1.7.1.

In the following we assume that $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ are all not zero subspace.

Part one

Suppose that $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$ is a direct sum.

Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r$ be a basis for $\mathbf{V}_1$, $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_s$ be a basis for $\mathbf{V}_2$, $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_t$ be a basis for $\mathbf{V}_3$. We show that $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r; \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_s; \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_t$ are linealy independent.

$\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r; \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_s; \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_t$ form a spanning set for $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$. We show that they are linearly dependent.

If $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r; \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_s; \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_t$ are linearly dependent, then there are constants $a_1, a_2, \cdots, a_r$, $b_1, b_2, \cdots, b_s$, $c_1, c_2, \cdots, c_s$, not all zero, such that

$$\mathbf{0}=(a_1\boldsymbol{\alpha}_1+a_2\boldsymbol{\alpha}_2+\cdots+a_r\boldsymbol{\alpha}_r)+(b_1\boldsymbol{\beta}_1+b_2\boldsymbol{\beta}_2+\cdots+b_s\boldsymbol{\beta}_s)+(c_1\boldsymbol{\gamma}_1+c_2\boldsymbol{\gamma}_2+\cdots+c_t\boldsymbol{\gamma}_t)$$

Among the three vectors

$$a_1\boldsymbol{\alpha}_1+a_2\boldsymbol{\alpha}_2+\cdots+a_r\boldsymbol{\alpha}_r,\quad b_1\boldsymbol{\beta}_1+b_2\boldsymbol{\beta}_2+\cdots+b_s\boldsymbol{\beta}_s,\quad c_1\boldsymbol{\gamma}_1+c_2\boldsymbol{\gamma}_2+\cdots+c_t\boldsymbol{\gamma}_t$$

there is at least one vector which is not zero. Thus, the zero vector has two distinct representations, which contradicts the assumption.

Hence, $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r; \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_s; \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots, \boldsymbol{\gamma}_t$ are linearly independent and span $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$. And

$\dim(\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3)=r+s+t=\dim(\mathbf{V}_1)+\dim(\mathbf{V}_2)+\dim(\mathbf{V}_3)$

Part two.

Suppose that $\dim(\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3)=\dim(\mathbf{V}_1)+\dim(\mathbf{V}_2)+\dim(\mathbf{V}_3)$. If there is a vector which has two distinct representations, $\mathbf{x}=\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3=\mathbf{w}_1+\mathbf{w}_2+\mathbf{w}_3$,

$\mathbf{0}=(\mathbf{u}_1-\mathbf{w}_1)+(\mathbf{u}_2-\mathbf{w}_2)+(\mathbf{u}_3-\mathbf{w}_3)$

$(\mathbf{u}_1-\mathbf{w}_1)$,$(\mathbf{u}_2-\mathbf{w}_2)$,$(\mathbf{u}_3-\mathbf{w}_3)$ are not all zero. Without loss of generality, we assume that $(\mathbf{u}_1-\mathbf{w}_1)\neq\mathbf{0}$, then there is a nonzero vector $(\mathbf{u}_1-\mathbf{w}_1)\in\mathbf{V}_1\cap(\mathbf{V}_2+\mathbf{V}_3)$. Thus,

$$\begin{aligned}\dim(\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3)&=\dim(\mathbf{V}_1)+\dim(\mathbf{V}_2+\mathbf{V}_3)-\dim(\mathbf{V}_1\cap(\mathbf{V}_2+\mathbf{V}_3))\\&<\dim(\mathbf{V}_1)+\dim(\mathbf{V}_2+\mathbf{V}_3)\leq\dim(\mathbf{V}_1)+\dim(\mathbf{V}_2)+\dim(\mathbf{V}_3)\end{aligned}$$

This contradicts the assumption.

第六题　Show that if $A \in C^{n\times n}$, then the column space of AA^H is the same as the column space of A. That is, $R(AA^H) = R(A)$.

Proof

$R(AA^H) = \{A(A^H\mathbf{x}) \mid \mathbf{x} \in C^n\}$, $R(A) = \{A\mathbf{x} \mid \mathbf{x} \in C^n\}$. Hence, $R(AA^H) \subset R(A)$.

Next we show that $R(AA^H) \supset R(A)$.

Suppose that $\mathbf{y} = A\mathbf{x} \in R(A)$, we want to show that $\mathbf{y} \in R(AA^H)$. That is to show that there is an $\mathbf{z} \in C^n$ such that $A\mathbf{x} = AA^H\mathbf{z}$. There is a $\mathbf{z} \in C^n$ such that $A^H\mathbf{z}$ equals the projection of $\mathbf{x}$ onto $R(A^H)$. Then $\mathbf{x} - A^H\mathbf{z} \perp R(A^H) = N(A)$. Thus, $A\mathbf{x} = AA^H\mathbf{z}$. This completes the proof of $R(AA^H) \supset R(A)$.

第七题　Let $A \in C^{n\times n}$. Show that if $A = QDQ^T$, where $Q \in R^{n\times n}$ is a real orthogonal matrix and $D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ with $|\lambda_k| = 1$ for $k = 1, 2, \cdots, n$, then A is both symmetric and unitary.

Proof

Since $A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A$, A is symmetric.

Since

$$
\begin{aligned}
& A^H A = (QDQ^T)^H QDQ^T = QD^H Q^T QDQ^T = QD^H DQ^T \\
& = Q\,\text{diag}(\overline{\lambda_1}, \overline{\lambda_2}, \cdots, \overline{\lambda_n})\,\text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) Q^T \\
& = Q\,\text{diag}(\overline{\lambda_1}\lambda_1, \overline{\lambda_2}\lambda_2, \cdots, \overline{\lambda_n}\lambda_n) Q^T = QIQ^T = QQ^T = I
\end{aligned}
$$

A is unitary.

第八题　Let $A \in C^{n\times n}$, and $AA^H = A^H A$. Show that $\|A\mathbf{x} - \lambda\mathbf{x}\| = \|A^H\mathbf{x} - \overline{\lambda}\mathbf{x}\|$ for any $\mathbf{x} \in C^n$ and $\lambda \in C$, where the inner product on C^n is the standard inner product.

$$
\begin{aligned}
& \|A\mathbf{x} - \lambda\mathbf{x}\|^2 = < A\mathbf{x} - \lambda\mathbf{x}, A\mathbf{x} - \lambda\mathbf{x} > = (A\mathbf{x} - \lambda\mathbf{x})^H (A\mathbf{x} - \lambda\mathbf{x}) \\
& = (\mathbf{x}^H A^H - \overline{\lambda}\mathbf{x}^H)(A\mathbf{x} - \lambda\mathbf{x}) \\
& = \mathbf{x}^H A^H A\mathbf{x} - \overline{\lambda}\mathbf{x}^H A\mathbf{x} - \mathbf{x}^H A^H(\lambda\mathbf{x}) + \overline{\lambda}\mathbf{x}^H\lambda\mathbf{x} \\
& = \mathbf{x}^H(A^H A)\mathbf{x} - \overline{\lambda}(\mathbf{x}^H A\mathbf{x}) - \lambda(\mathbf{x}^H A^H\mathbf{x}) + \lambda\overline{\lambda}(\mathbf{x}^H\mathbf{x})
\end{aligned}
$$

$$
\begin{aligned}
& \|A^H\mathbf{x} - \overline{\lambda}\mathbf{x}\|^2 = < A^H\mathbf{x} - \overline{\lambda}\mathbf{x}, A^H\mathbf{x} - \overline{\lambda}\mathbf{x} > = (A^H\mathbf{x} - \overline{\lambda}\mathbf{x})^H (A^H\mathbf{x} - \overline{\lambda}\mathbf{x}) \\
& = (\mathbf{x}^H A - \lambda\mathbf{x}^H)(A^H\mathbf{x} - \overline{\lambda}\mathbf{x}) \\
& = \mathbf{x}^H AA^H\mathbf{x} - \lambda\mathbf{x}^H A^H\mathbf{x} - \mathbf{x}^H A(\overline{\lambda}\mathbf{x}) + \lambda\mathbf{x}^H\overline{\lambda}\mathbf{x} \\
& = \mathbf{x}^H(AA^H)\mathbf{x} - \overline{\lambda}(\mathbf{x}^H A\mathbf{x}) - \lambda(\mathbf{x}^H A^H\mathbf{x}) + \lambda\overline{\lambda}(\mathbf{x}^H\mathbf{x})
\end{aligned}
$$

Since $AA^H = A^H A$, we obtain that $\|A\mathbf{x} - \lambda\mathbf{x}\| = \|A^H\mathbf{x} - \overline{\lambda}\mathbf{x}\|$